Physikalisches Praktikum II im SS 2013 Universität Stuttgart

Report for experiment V15: Quantum Analogs

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Summary

As the name of this experiment already states, we will investigate analogies to quantum mechanical phenomena. In this special case we are about to compare resonance phenomena based on acoustics with certain quantum mechanical models. Therefore we induce stationary waves in pipes and measure their resonance behaviour.

Different combinations of pipes are set up to simulate different quantum mechanical problems. Here we choose one pipe closed at both ends to imitate the infinite potential well. A spherical resonator is used to mimic the spherical harmonics which are also found in the hydrogen atom. A set of pipes with apertures in periodical intervals represents a naive model of a one dimensional solid state body. We want to extract its behaviour in regard of band structure and its dispersion relation.

The spectra of the above variations of pipes are measured and analysed by a computer.

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1 Basics and Theoretical Background

In this section we discuss the physical background of the experiment. As we investigate the analogies of acoustics and quantum mechanics respectively solid state physics we need to review the basics of both fields. We begin with the quantum mechanics part.

1.1 Wave Function and Schrödinger's equation

The mathematical basis of quantum mechanics is the Hilbert space \mathcal{H} . On this Hilbert space the states of our system are living in form of vectors, which are denoted using the Dirac notation. As looking at states in an abstract mathematical space is way too difficult for physicists, we transfer the states to a stationary base, by using the inner product of the Hilbert space \mathcal{H} and transforming to another notation.

$$\langle x|\psi\rangle = \psi(x) \tag{1}$$

We normally deal with states in \mathbb{R}^3 , which are additionally time dependent. Therefore it is better to write

$$\langle \boldsymbol{r}|\psi(t)\rangle = \psi(\boldsymbol{r},t).$$
 (2)

Now we want to describe the dynamics of this complex wave function. This is described by one of the postulates of quantum mechanics: *Schrödinger's equation*. As already said Schrödinger's equation is a postulate which means neither does it introduce any new quantity nor can it be derived from any other definitions. Although it shows some similarities with the classical wave equation.

$$i\hbar \frac{d}{dt}\psi(\mathbf{r},t) = \hat{H}\psi(\mathbf{r},t), \qquad (3)$$

is Schrödinger's equation in a stationary, three dimensional basis.

In solids the Hamiltonian \hat{H} is time independent, so Schrödinger's equation for an electron in a solid with the binding potential $V(\mathbf{r})$ reads

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\psi(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})\right]\psi(\mathbf{r},t).$$
(4)

In this case we can separate the time dependent part from the rest, by using the *unitary time* evolution operator $U(t, t_0) = \exp[-\frac{i}{\hbar}\hat{H}(t - t_0)]$. With the ansatz $\psi(\mathbf{r}, t) = U(t, t_0)\phi(\mathbf{r})$ we get Schrödinger's time independent equation

$$E\phi(\mathbf{r}) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})\right]\phi(\mathbf{r}),\tag{5}$$

where *E* are the energy eigenvalues of the Hamiltonian.

1.1.1 The Infinite Potential Well

The infinite potential well a one dimensional eigenvalue problem, which is described by the potential

$$V(x) = \begin{cases} \infty & \text{if } |x| > a \\ 0 & \text{else} \end{cases}.$$
(6)

A schematic plot of the potential can be seen in figure 1.

1 Basics

We write down Schrödinger's time independent equation for the part where V(x) = 0, i.e. |x| < a. Anywhere else in \mathbb{R}^3 there can't exist a wave function, due to the potential being $V(|x| > a) = \infty$.

$$E\phi(x) = -\frac{\hbar^2}{2m}\partial_x^2\phi(x) \tag{7}$$

As one can easily see from figure 1 we get the boundary conditions $\phi(-a) = \phi(a) = 0$. Also, because the potential is symmetric, we can assume (7) to have symmetric and antisymmetric solutions, what gives us the boundary conditions $\phi(x) = \phi(-x)$ in the symmetric case and $\phi(x) = -\phi(-x)$ in the antisymmetric case.



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FIGURE 1: The infinite potential well of width 2*a*. Plotted are the first three eigenfunctions at the corresponding energy level.

As ansatz we choose a real function in form of a linear superposition of sine and cosine

$$\phi(x) = c_1 \cos(kx) + c_2 \sin(kx), \qquad k = \sqrt{\frac{2mE}{\hbar^2}},$$
(8)

which decomposes to a symmetric and an antisymmetric solution, where

$$\phi_S(x) = c_1 \cos(kx),$$

$$\phi_A(x) = c_2 \sin(kx).$$
(9)

Applying the boundary condition $\phi(-a) = \phi(a) = 0$ to these solutions gives us

$$k_{S} = \frac{(2n-1)\pi}{2a} \\ k_{A} = \frac{(2n)\pi}{2a}$$
 $n \in \mathbb{N}.$ (10)

So the complete solution for the infinite potential well is

$$\phi(x) = \begin{cases} \frac{1}{\sqrt{a}} \cos\left(\frac{n\pi x}{2a}\right) & n = 1, 3, 5, \dots \\ \frac{1}{\sqrt{a}} \sin\left(\frac{n\pi x}{2a}\right) & n = 2, 4, 6, \dots \end{cases}$$
(11)

with the eigenenergies

$$E_n = \frac{\hbar^2}{2m} \left(\frac{\pi}{2a}\right)^2 n^2, \qquad n \in \mathbb{N}.$$
 (12)

From (8) we directly get the dispersion relation

$$E(k) = \frac{\hbar^2}{2m}k^2, \qquad k = \frac{n\pi}{2a}, n \in \mathbb{N}.$$
(13)

If $|a| \to \infty$, *k* becomes an infinitesimal fraction and can therefore be picked from \mathbb{R} . We now have a free electron with continuous spectrum.

1.1.2 The Hydrogen Atom

The hydrogen atom consists of a proton at the centre orbited by an electron. Like in the classical mechanics central potential problem, the centre-of-momentum motion can be separated from the relative motion of the electron.



FIGURE 2: A model for the hydrogen atom, where p denotes the proton and e^- the electron. Also drawn are the spherical coordinates used for the solution of the problem.

The interaction of the proton and the electron is given by the Coulomb potential

$$V(r) = \frac{Ze^2}{r}.$$
 (14)

It involves the atomic number Z which in this case is 1. After transformation to coordinates in the centre-of-mass system, Schrödinger's equation for the electron reads

$$\left[-\frac{\hbar^2}{2m_e}\nabla^2 - \frac{Ze^2}{r}\right]\psi(\mathbf{r}) = E\,\psi(\mathbf{r}).\tag{15}$$

As one can see from figure 2 spherical coordinates are best for solving this problem. Using the ansatz

 $\psi(r,\theta,\varphi) = R_{n,\ell}(r)Y_{\ell,m}(\theta,\varphi)$ the problem splits into a radial part and an angular part. Using the Laplacian operator in spherical coordinates, we get

$$\left[-\frac{\hbar^2}{2m_e r}\frac{\partial^2}{\partial r^2}r - \frac{\hbar^2\ell(\ell+1)}{2m_e r^2} - \frac{e^2}{r}\right]R_{n,\ell}(r) = ER_{n,\ell}(r),$$
(16)

$$-\hbar^{2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^{2}\theta}\frac{\partial^{2}}{\partial\varphi^{2}}\right]Y_{\ell,m}(\theta,\varphi) = \hbar^{2}\ell(\ell+1)Y_{\ell,m}(\theta,\varphi).$$
(17)

The solutions of these equations are nontrivial.

Solving the radial part gives us

$$R_{n,\ell} = \sqrt{\left(\frac{2}{na_0}\right)^3 \frac{(n-\ell-1)!}{2n(n+\ell)!}} \exp\left(-\frac{r}{na_0}\right) \left(\frac{2r}{na_0}\right)^\ell \mathcal{L}_{n-\ell-1}^{2\ell+1}\left(\frac{2r}{na_0}\right),$$
(18)

where a_0 is Bohr's radius and the $\mathcal{L}_n^{\ell}(r)$ are the *associated Laguerre polynomials*.

$$\mathcal{L}_{n}^{\ell}(r) = \frac{r^{-\ell}}{n!} \exp(r) \frac{\mathrm{d}^{n}}{\mathrm{d}r^{n}} \Big(\exp(r) r^{n+\ell} \Big).$$
(19)

The solution of the angular part is given by the *spherical harmonics* $Y_{\ell,m}(\theta, \varphi)$,

$$Y_{\ell,m}(\theta,\varphi) = \sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}} (-1)^m \exp(-\mathrm{i}m\varphi) \,\mathcal{P}_{\ell}^m(\cos\theta),\tag{20}$$

where \mathcal{P}_{ℓ}^{m} are the *associated Legendre polynomials*.

$$\mathcal{P}_{\ell}^{m}(x) = \frac{(-1)^{\ell}}{2^{\ell}\ell!} (1 - x^{2})^{m/2} \frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}} (x^{2} - 1)^{\ell}.$$
(21)

The eigenergies can be determined by either using the above solutions in Schrödinger's equation or by introducing super symmetric ladder operators, which is the more beautiful ansatz, see [5, pp. 113–117].

$$E_n = -\frac{Z^2 e^4 m_e^2}{2\hbar} \frac{1}{n^2} , \qquad n \in \mathbb{N}.$$
 (22)

Taking into account, that we set $4\pi\varepsilon_0 \equiv 1$, we get a ground state energy E_1 of

$$E_1 = -13.6 \,\mathrm{eV}$$
 (23)

As seen above, the hydrogen atom is described by the quantum numbers n, ℓ and m, where n is called the principal quantum number, ℓ is the angular momentum quantum number and m is the magnetic quantum number. These numbers follow certain rules, such as

$$n \in \mathbb{N}$$
, (24)

$$\ell \in \{0, \dots, n-1\},\tag{25}$$

$$m \in \{-\ell, \dots, 0, \dots, \ell\}.$$

$$(26)$$

In the normal hydrogen atom, the states with the same quantum number ℓ , but different *m* have the same energy level, so they can't be distinguished. This phenomenon is known as *m*-degeneracy. This degeneracy can be lifted if the spin-orbit-coupling is taken into account. To lift the degeneracy in the kind that one can actually measure the split up of the energy level, a perturbation has to be applied, normally in the form of an external magnetic field, which couples with the spin. This is know as the *Zeeman-effect*, see [1, p. 360].

1.1.3 The Hydrogen Molecule

The lonized Hydrogen Molecule The hydrogen molecule ion H_2^+ consists of an electron bound between two protons. To approximate this problem, we can modify Schrödinger's equation of the Hydrogen atom with an additional potential, that considers the interaction of the electron with the second proton. Physically this can be interpreted as moving the second proton from a distance, where the interaction is negligible to the vicinity of the atom. The equation then takes the form

$$-\frac{\hbar^2}{2m_e}\nabla^2 - \frac{Ze^2}{r_a} - \frac{Ze^2}{r_b} \psi(\mathbf{r}) = E\psi(\mathbf{r}).$$
⁽²⁷⁾

with r_a and r_b are the distances of the electron to either proton. A solution can be obtained by a superposition of wave functions for the interaction of the electron with each proton. The resulting wave functions are not orthogonal, so the combination of solutions results in bonding and antibonding orbitals.

The Hydrogen Molecule In the hydrogen molecule we have two protons and two electrons. The electrons will carry the indices 1 and 2, while the protons receive the indices *A* and *B*. The combinations 1-A and 2-B form one single hydrogen atom, each. Applying the Born-Oppenheimer-Approximation we ignore the dynamics of the centre of mass and drop the terms describing the movement of the protons. The Hamiltonian can be derived as a composition of two hydrogen atoms with an additional interaction term. To get a better overview figure 3 yields a sketch of the different positions and distances in the system.

$$H = H_1 + H_2 + V_{\text{int}}$$
(28)

$$= \left\{ -\frac{n}{2m} \nabla_{1}^{2} - \frac{e^{-}}{|\mathbf{r}_{1} - \mathbf{r}_{A}|} \right\} + \left\{ -\frac{n}{2m} \nabla_{2}^{2} - \frac{e^{-}}{|\mathbf{r}_{2} - \mathbf{R}_{B}|} \right\} + \frac{e^{2}}{|\mathbf{r}_{1} - \mathbf{r}_{2}|} + \frac{e^{2}}{|\mathbf{R}_{A} - \mathbf{R}_{B}|} - \frac{e^{2}}{|\mathbf{r}_{1} - \mathbf{R}_{B}|} - \frac{e^{2}}{|\mathbf{r}_{2} - \mathbf{R}_{A}|}$$
(29)

Schrödinger's equation for this Hamiltonian can be solved using perturbation theory with an approximation of V_{int} as the perturbing Hamiltonian. Unperturbed eigenstates are given by a linear combination of product states $\psi = \psi_1 \psi_2$ of the composite hydrogen atoms. As in the case of H_2^+ the resulting wave functions can be classified as bonding and antibonding, but as there are two electrons not every possible solution is physically correct. Since electrons are fermions, we have to take Pauli's principle into account. According to Pauli's principle no more than one

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FIGURE 3: A model for the hydrogen molecule, from [1, p. 372, fig. 14.5]. The positions of the electrons are denoted by r_1 and r_2 , while the positions of the protons are described with R_A and R_B . The distance vector between the two protons is R.

fermion can populate the same quantum state. Therefore the fermionic wave function has to be antisymmetric in every coordinate, which also includes the spin of the electrons. The possible combinations are

$$\psi_1(\mathbf{r}_1, \mathbf{r}_2) = \chi_{\uparrow}(1)\chi_{\uparrow}(2)[\psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2) - \psi_1(\mathbf{r}_2)\psi_2(\mathbf{r}_1)]$$
(30)

$$\psi_2(\mathbf{r}_1, \mathbf{r}_2) = \chi_{\downarrow}(1)\chi_{\downarrow}(2)[\psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2) - \psi_1(\mathbf{r}_2)\psi_2(\mathbf{r}_1)]$$
(31)

$$\psi_3(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}} [\chi_{\uparrow}(1)\chi_{\downarrow}(2) + \chi_{\uparrow}(2)\chi_{\downarrow}(1)] [\psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2) - \psi_1(\mathbf{r}_2)\psi_2(\mathbf{r}_1)]$$
(32)

$$\psi_4(\mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}} [\chi_{\uparrow}(1)\chi_{\downarrow}(2) - \chi_{\uparrow}(2)\chi_{\downarrow}(1)] [\psi_1(\mathbf{r}_1)\psi_2(\mathbf{r}_2) + \psi_1(\mathbf{r}_2)\psi_2(\mathbf{r}_1)]$$
(33)

where χ_{\uparrow} is the spin-up and χ_{\downarrow} the spin-down wave function. The first three functions belong to triplet states, while the latter belongs to a singlet state. From these equations we see, that the representation of molecular orbitals as linear combinations of atomic orbitals (LCAO) is a possible approximation. The LCAO method can also explain, why certain types of molecular bondings such as He₂ aren't possible. This is due to the electron occupancy of bonding and antibonding orbitals. An example LCAO electron occupancy scheme of the H₂ ground state is depicted in figure 4. The quantum number *m* is carried over from the atomic case and is used to differentiate orbitals in terms of their symmetry axis or plane. In analogy to the labelling of the quantum number ℓ the molecular orbitals are named σ , π and δ . The σ -orbitals have a symmetry axis in the direction of the connecting line between the atoms while π -orbitals have a symmetry plane perpendicular to this line intersecting it at its midpoint.



FIGURE 4: A sketch of the electron distribution in the molecular orbitals of the H₂ molecule. The lines denoted with 1s are the atomic orbitals of the hydrogen atoms, σ is the bonding molecular orbital and σ^* is the antibonding molecular orbital.

1.1.4 Lifetime of States

In general eigenstates of a quantum mechanical system that are not equivalent to the ground state have a finite lifetime. This is due to the fact that thermodynamics also applies for quantum mechanical systems. Therefore, also in these cases dissipation of energy occurs. The differential equations of simple quantum dynamics are not intended for this. Therefore so called *Master*

1 Basics

equations are derived to describe the incoherent decays using Lindblad operators A_k .

$$\mathcal{L}\varrho = -i[\hat{H},\varrho] + \sum_{k=1}^{N^2 - 1} \gamma_k \left(A_k \varrho A_k^{\dagger} - \frac{1}{2} A_k^{\dagger} A_k \varrho - \frac{1}{2} \varrho A_k^{\dagger} A_k \right)$$
(34)

where \mathcal{L} is the Lindblad generator, γ_k are correlation functions of the environment and ϱ is the systems density matrix.

If we are not interested in the process of the decay, we can just assume a non-unitary time evolution to describe the propagation of the system afterwards. Therefore an additional decay factor is added to the normal unitary time-evolution, namely

$$U(t,t_0) = \exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}t\right) \quad \rightsquigarrow \quad \mathcal{D}(t,t_0) = \exp\left(-\lambda t - \frac{\mathrm{i}}{\hbar}\hat{H}t\right) \tag{35}$$

where $\lambda > 0$. Non-coherence is implied by the fact, that the absolute squared of a wave function propagated using this time evolution is $\neq 1$.

$$|\psi\rangle = \mathcal{D}(t, t_0) |\phi\rangle = \exp\left(-\lambda t - \frac{\mathbf{i}}{\hbar}\hat{H}t\right) |\phi\rangle$$
 (36)

$$\| |\psi\rangle \|^2 = \langle \psi |\psi\rangle = \exp\left(-2\lambda t\right) \to 0 \quad \text{for } t \to \infty$$
(37)

We see that for large time spans the state vanishes; it can't even be populated any more, as it is gone.

This process of vanishing leads to a spectral line broadening known from spectroscopy of atomic transitions. The line broadening can be of several artificial origins, like coupled electric or magnetic fields or a heat bath. There also exists a natural broadening due to the uncertainty principle. It can be explained as follows:

Consider the energy-time-uncertainty-principle

$$\Delta E \cdot \Delta t \gtrsim h$$

This means the energy can be measured sharper, the longer the corresponding state lives. Due to the finite lifetime the energy gathers uncertainty and because the energy is expressed through the frequency of the photons emitted ($E = h \cdot v$) the spectral line becomes diffuse.

For a mathematical derivation consider a plane wave function with finite lifetime

$$\psi(t) = \mathrm{e}^{-\lambda t - \mathrm{i}\omega_0 t}$$

Now we want to calculate the probability of finding ψ in the state $\phi(t) = e^{-i\omega t}$. Therefore we write

$$A(\omega) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \phi^*(t) \,\psi(t) \,\mathrm{d}t \tag{38}$$

$$=\frac{1}{\sqrt{2\pi}}\int_{0}^{\infty}e^{i\omega t}e^{-\lambda t-i\omega_{0}t}dt$$
(39)

$$=\frac{1}{\sqrt{2\pi}}\frac{1}{\lambda+i(\omega_0-\omega)}\tag{40}$$

If we square $A(\omega)$ we get a so called Lorentzian peak

$$||A(\omega)||^{2} = \frac{1}{2\pi} \frac{1}{(\omega_{0} - \omega)^{2} + \lambda^{2}}$$
(41)

The lifetime τ of the state is just the HWHM (Half Width at Half Maximum) of that peak, that means the time after the peak decayed to 1/e. This implies, that the energy width Γ of the state is given through

$$\Gamma = \frac{\hbar}{\tau} \tag{42}$$

1.2 Solid State Physics

Condensed matter in the solid state often establishes crystal structures. An ideal crystal can be characterized as a periodic lattice with a basis of atoms on every lattice point. It is possible to address every lattice point r with integer coordinates and a distinctive basis of lattice vectors, such as

$$\mathbf{r} = \mathbf{r}' + u_1 \mathbf{a}_1 + u_2 \mathbf{a}_2 + u_3 \mathbf{a}_3 \tag{43}$$

where the a_i are the basis vectors and the u_i are integer coordinates. The parallelepiped constituted by such a basis is called a primitive cell. There can be other periodic structures in a lattice, like variations of a cubic structure that have additional lattice points, either in the centre of the cube or on the cube's faces. Another example of a primitive cell is the Wigner-Seitz cell. It can be constructed by connecting a lattice point to all of its closest nearby lattice points and drawing perpendicular lines at each midpoint of the connecting lines. The construction is shown in figure 5.





Because of the periodicity, every lattice point is invariant under translation by any given lattice vector. Thus every physical property associated with a lattice point, such as the electron density, has the same translational invariance. A physical effect that is caused by this periodicity is x-ray diffraction. Because the wavelength λ of x-radiation is of the same magnitude as the lattice constant *d*, the scattering of the beam on several parallel planes, separated by the distance *d*, causes constructive interference for a certain angle θ between the direction of the beam and the orientation of the planes. Mathematically this is expressed by Bragg's law

$$n\lambda = 2d\sin\theta \tag{44}$$

where *n* is an integer and $n\lambda$ is the difference of wavelengths needed for constructive interference. With this effect it is possible to analyse the structure of a crystal lattice. In order to obtain information about the atomic base structure and the distribution of electrons, we additionally need to calculate the intensity of the scattered beams. If we assume that the intensity is proportional to the electron density itself, a Fourier-analysis of the periodic properties leads to useful results. This method leads to the concept of reciprocal space, which consists of all vectors *G* of the Fourier series of the electron density

$$n(\mathbf{r}) = \sum_{\mathbf{G}} n_{\mathbf{G}} \exp\left(i\mathbf{G} \cdot \mathbf{r}\right) \tag{45}$$

where the equation is invariant under translation of the lattice. The reciprocal vectors define a reciprocal lattice. It can be shown that the difference Δk of wave vectors of the diffracted and incident beams of a wave fulfilling Bragg's law is a reciprocal lattice vector. Therefore

$$k\frac{G}{2} = \left(\frac{G}{2}\right)^2 \tag{46}$$

is another form of Bragg's law.

The Wigner-Seitz cell of the reciprocal lattice is called the first Brillouin zone. Looking at the construction method for a Wigner-Seitz cell we see that the perpendicular planes enclosing the Brillouin zone are the endpoints of possible wave vectors fulfilling equation (46). Extending the construction method by connecting the original lattice point to every other point of the reciprocal lattice we get additional wave vectors.

1.2.1 Band Gap

The band gap is another effect that stems from the periodicity of the lattice. By introducing a periodic potential in the Schrödinger equation of a free electron the hamiltonian and the displacement operator have shared eigenstates. This results in Bloch's theorem which say that every solution of Schrödinger's equation with a periodic potential takes the form of

$$\psi_k = u_k(\mathbf{r}) \exp\left(i\mathbf{k}\mathbf{r}\right) \tag{47}$$

with a periodic function u_k . We can use Bloch's theorem to determine the possible wave vectors in the Kronig-Penney model of a one dimensional periodic delta potential

$$V(x) = V_0 \sum_{j} \delta(x - ja)$$
(48)

where *a* is the distance between two peaks. The wave function and its derivative have to be continuous. Utilizing this condition and integrating Schrödinger's equation we get several equations that can be combined to

$$\cos(ka) = \cos(Ka) + \frac{A}{Ka}\sin(Ka)$$
(49)

with a positive constant A. According to this equation every wave number k of the electron corresponds to a certain energy E(K). Not every choice of K is possible, which leads to energy or band gaps in the dispersion relation. The dispersion relation can also be depicted in the first Brillouin zone since the periodicity causes a translation invariance by a vector of the reciprocal lattice. For a more realistic potential similar results can be derived by expanding the potential and the wave function to a Fourier series and inserting them in Schrödinger's equation. The band gaps always occur for a wave vector on the border of a Brillouin zone since Bragg's law applies in this case. The wave vector is then changed by a vector of the reciprocal lattice and the dispersion relation can't be continuous for this wave vector.

1.3 Analogies between Quantum Mechanics and Acoustics

This part investigates the concrete analogies between quantum mechanics and acoustics. Furthermore the theory of the experiments performed are discussed.

1.3.1 The Wave Equation

The dynamics of every flowing material can be described by the Navier-Stokes-equation

$$\varrho\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla\right)\boldsymbol{u} = -\nabla p + \nabla \cdot \mathsf{T} + f \tag{50}$$

where ρ is density, *u* is the flow velocity, *p* is the pressure, T is the stress tensor and *f* are arbitrary external forces on the fluid.

In principle this is nothing else than conservation of momentum in continuum.

Assuming a viscosity of $\eta = 0$, a laminar flow profile and a constant density ϱ the Navier-Stokes-equations reduce to the *Euler-equations*

$$\left(\frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla\right) \boldsymbol{u} + \frac{1}{\varrho} \nabla \boldsymbol{p} = 0 \tag{51}$$

If we now linearise this equation in the flow velocity u we obtain

$$\frac{\partial u}{\partial t} + \frac{1}{\varrho} \nabla p = 0 \tag{52}$$

We find the density again in the continuity equation, where

$$\frac{\partial}{\partial t}\varrho = -\nabla \cdot \boldsymbol{j} = -\varrho \nabla \cdot \boldsymbol{u}$$
(53)

This equation is the mathematical formulation of the philosophical statement "Nothing comes from nothing". In this case it describes that the fluid can neither vanish nor be created in any place. One also has

$$\frac{\partial p}{\partial \varrho} = \frac{1}{\kappa \varrho} \tag{54}$$

where κ is the compressibility. All these together and further linearisations lead to

$$\frac{1}{\kappa\varrho}\nabla^2 p - \frac{\partial^2 p}{\partial t^2} = 0 \tag{55}$$

where $c = 1/\sqrt{\kappa \varrho}$ is the speed of sound.

1.3.2 The Infinite Potential Well

To model an infinite potential well we consider a tube with sealed ends. In this tube stationary waves are induced. Therefore the sound has to interfere with the right phase, i.e. the wave function has to fulfil the Neumann boundary conditions

$$\left. \frac{\partial u}{\partial t} \right|_{z=0} = 0 \qquad \left. \frac{\partial p}{\partial z} \right|_{z=0} = 0 \tag{56}$$

Thus it follows that the possible wavelengths for standing waves have to obey

$$2L = n\lambda = n\frac{c}{\nu} \tag{57}$$

Now we want to get a standing wave in the tube. Therefore we make the ansatz

$$p(x,t) = \hat{p}\cos(kx - \omega t + \phi_0)$$
(58)

with the wave number k, the vibration frequency ω and an additional phase factor ϕ_0 .

To get a standing wave the superposition of a left and a right oriented wave is needed, see

$$p(x,t) = \frac{1}{2}\cos(kx - \omega t + \phi_0) + \frac{1}{2}\cos(-kx - \omega t + \phi_0)$$
(59)

$$=\cos(kx+\phi_0)\cos(\omega t) \tag{60}$$

Again using the Neumann boundary conditions we get for the parameters

$$k = \frac{n\pi}{L} \qquad \phi_0 = 0 \tag{61}$$

1 Basics

When we now solve the acoustic wave equation for the conditions we obtain a result, that is pretty similar to the one seen in quantum mechanics. For the quantum mechanical case we consider a particle of $E = \hbar \omega$ in an infinite potential well in the interval $x \in [0, L]$.

$$\psi(x,t) = \sqrt{\frac{2}{L}} \exp\left(-i\omega t\right) \sin\left(\frac{n\pi}{L}x\right) \qquad \longleftrightarrow \qquad p(x,t) = \hat{p}\cos\left(\omega t\right) \cos\left(\frac{n\pi}{L}x\right) \tag{62}$$

The dispersion relation reads

$$E(k) = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2}{2mL^2} n^2 \qquad \iff \qquad \omega(k) = c k = \frac{\pi c}{L} n \tag{63}$$

We can now break down all the facts above to the core differences between the quantum mechanical and the acoustic wave equation.

- 1. The time derivative: In the acoustic wave equation we gain periodicity in the solutions through a second order time derivative, while in the equantum mechanical case the periodicity is ensured be the complex phase factor i.
- 2. Standing waves: In both cases standing waves are noticeable, which are harmonic in time and space. The main difference is that the quantum mechanical wave equation is not real and can not be measured at all. Only their squared absolute can be measured and also only in terms of probability experiments.
- 3. Phase: Directly inherited from the point before is the fact that the phase of the quantum mechanical wave function can not be measured, while is possible to measured the phase of an acoustic wave using a usual in trade oscilloscope.
- 4. The boundary conditions: In quantum mechanics the wave function at the boundary is given by the Dirichlet boundary condition, which ensures that the wave function decays to zero at these points. In the acoustic wave equation these boundary conditions are the Neumann conditions, which specifies the values of the derivative of the function at the edges.
- 5. The dispersion relation: As seen above the dispersion relation for acoustic waves is linear, where the one for quantum particles is parabolic.
- 6. The energy: The eigenvalues of the Hamiltonian are considered the energy of a quantum mechanical wave, where in acoustics the energy of a wave is calculated more classical

$$E = E_{\text{pot}} + E_{\text{kin}} = \int_{V} \frac{p^2}{2\varrho_0 c^2} dV + \int_{V} \frac{\varrho v^2}{2} dV$$
(64)

1.3.3 Line Broadening

In the classical case of the damped, driven harmonic oscillator we find the following differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x + 2\gamma \frac{\mathrm{d}}{\mathrm{d}t}x + \omega_0^2 x = K \mathrm{e}^{\mathrm{i}\omega t} \tag{65}$$

In our case of acoustic experiments, we can view the oscillator as the standing sound wave and the driving force as the speaker.

We find a dependency of the amplitude to the driving frequency ω . As a solution to the differential equation a superposition of a homogeneous and steady-state solution can be found

$$p(t) = A_1 e^{-\gamma t} \cos(\omega_1 t + \varphi_1) + A \cos(\omega t + \varphi)$$
(66)

1 Basics

with the damped resonance frequency ω_1 and the driving frequency ω .

We assume the transient solution to have damped out, so only the driven term is detected. Using the complex notation of the cosine function the amplitude then reads

$$|A| = \left| \frac{K e^{i\varphi}}{(\omega_0^2 - \omega^2) + (2i\gamma\omega)} \right| = \frac{K}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}}$$
(67)

In reality not only a single resonance is excited at once, but many more. Therefore we will have to write the amplitude as a coherent superposition of multiple amplitudes

$$|A(\omega)| = \left|\sum_{n=1}^{N} A_n(\omega)\right| = \left|\sum_{n=1}^{N} \frac{K_n e^{i\varphi_n}}{(\omega_n^2 - \omega^2) + (2i\gamma_n\omega)}\right|$$
(68)

Using this notation we obtain the fitting parameters K_n , φ_n , ω_n and γ_n for each peak in our spectrum.

Last we compare the energy broadening of a state with finite lifetime to the classical case (right) and find

$$|A(\omega)| = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + \lambda^2}} \qquad \iff \qquad |A(\omega)| = \frac{K}{\sqrt{(\omega_0^2 - \omega^2)^2 + (2\gamma\omega)^2}} \tag{69}$$

We notice that these are nearly similar.

Concluding the decay of the amplitude of a classical driven, damped harmonic oscillator is a reasonable model for the line broadening, seen in quantum mechanics.

1.3.4 The Hydrogen Atom

Let's consider a wave in a spherical resonator. The wave equation is the common one

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \nabla^2 p \tag{70}$$

We will only take standing waves into account and as seen above the time dependency can be separated, like $\tilde{p}(\mathbf{r}, t) = p(\mathbf{r}) \cos(\omega t)$. Plugging this in the wave equation

$$-\frac{\omega^2}{c^2}p(\mathbf{r})\cos(\omega t) = \nabla^2 p(\mathbf{r})\cos(\omega t)$$
(71)

A transformation to spherical coordinates allows us to separated radial and angular parts, where we write $p(\mathbf{r}) = R(r) Y(\theta, \varphi)$

$$-\frac{\omega^2}{c^2}R(r)Y(\theta,\varphi) = \left[\Delta_r + \frac{1}{r^2}\Delta_{\theta,\varphi}\right]R(r)Y(\theta,\varphi)$$
(72)

This leads to two equations

$$\left[\frac{\partial^2}{\partial r^2}r + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2}\right]R(r) = -\frac{\omega^2}{c^2}R(r),$$
(73)

$$-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]Y_{\ell,m}(\theta,\varphi) = \ell(\ell+1)Y_{\ell,m}(\theta,\varphi).$$
(74)

It is easy to see that the angular part of the spherical resonator is completely similar to the one of the hydrogen atom, while the radial part is given by the *Bessel-functions* in the acoustic case. If we look at the spherical infinite potential well the analogy would go even further as the radial part

would also be the same. In the classical sense we can also describe the angular resonances by quantum numbers n', ℓ and m with the difference that the quantum mechanical ℓ degeneracy isn't given here, although the m degeneracy still exists and can be observed when breaking the symmetry of the spherical resonator. The energy is in the quantum mechanical case given by the eigenvalues of the Hamiltonian, while in the acoustic case the energy is computed with (64).

Perturbed Resonator

By introducing a spacer ring between the two hemispheres of the spherical resonator, the symmetry is broken and the quantization axis no longer points from the microphone through the speaker, but along the symmetry axis of the stretched resonator. For all the resonances with quantum number ℓ , which are $2\ell + 1$ fold degenerate, the $m \in [-\ell, ..., 0, ..., \ell]$ modes are also excited, see also figure 6.



FIGURE 6: Sketch for the perturbation, from [4, p. 3-2]. From left to right: 1.) The unperturbed spherical resonator, where the quantization axis points through speaker and microphone. 2.)–4.) The quantization axis now points through the symmetry axis of the elongated resonator. Higher m modes are also excited.

The degeneracy can not be lifted completely as in the experiment the sign of $\pm m$ can not be determined, as they both get excited by the speaker and have the same amplitude. The only difference is their relative phase, as the one is circulating around the quantization axis clockwise and the other anticlockwise. A superposition of two spherical harmonics with the different phase factors $\pm \varphi$ leads to a standing wave as in

$$e^{im\varphi} + e^{-im\varphi} = 2\cos(m\varphi) \tag{75}$$

The behaviour of lifting the degeneracy through breaking the symmetry can be compared to the quantum mechanical *Zeeman-effect*, where an external magnetic field is coupled to a hydrogen atom. Because the quantization axis is in that case given through the magnetic field it has an explicit orientation, such that the $\pm m$ degeneracy is lifted as well.

1.3.5 The Hydrogen Molecule

As in the case of perturbation theory, modeling an H_2 molecule can be achieved by increasing the coupling between two Hydrogen atoms. In the acoustic case the two spherical resonators are coupled by an iris along the z-axis. By increasing the iris diameter we should also increase the coupling strength. Since the position of the microphone can only be varied in the *x*, *y*-plane on the upper resonator, the shape of higher orbitals can only be partly analysed. We will only find varying amplitudes for π -orbitals for different angles α .

1.3.6 A Solid

A solid is considered to be a lattice, which is especially true in case of a crystal. We now assume that the crystal is of infinite size. Then our wave number k is no longer quantized, but continuous

1 Basics

and the dispersion relation reads

$$E(k) = \frac{\hbar^2 k^2}{2m} \tag{76}$$

From solid state physics we know that electrons scatter near the edges of the Brillouin zone, i.e. if $\lambda = 2a$.

In the acoustic case we find a linear dispersion relation, which is proportional to the speed of sound

$$\omega(k) = \frac{c}{2\pi}k\tag{77}$$

The solid is modelled by a tube with intermittent iridae, to simulate the atomic cores. We expect, that periodicity in the region of the half wavelength leads to changes in the dispersion relation. If the sound is induced in the iridae mounted tube a band structure should form. The amount of resonance peaks per band should be proportional to the number of elementary cells in the tube.

In terms of solid state physics there are two possible explanations for the forming of the same band structure:

The Periodic Potential To model this case the spectrum of an empty tube is recorded and then iridae are put in between the elementary cells. The iris diameter is a scale for the binding strength of the scattering points.

The Tight Binding Theory To simulate the formation of bonds between an increasing number of atoms, we connect the parts of a resonator tube piecewise with intermittent iridae. The formation of a concrete band structure can be observed when plugging in more and more pieces.

To produce a defect in the tube, there are different options. We could either vary the distance between two iridae, which would correspond to a lattice, where one of the elementary cells possesses another lattice constant, than the others or we could vary the diameter of the iridae, which would correspond to a doped lattice, where atoms with different binding potentials are present.

2 Analysis and Interpretation

2.1 Tube Resonator as a Model for the Infinite Potential Well

Procedure We observe the spectrum of a tube resonator as a model for a one dimensional infinite potential well. We will measure spectra of differently sized tubes, where the tube size is either given by L = 600 mm or by L = 150 mm.

Therefore in both the end caps are a speaker and a microphone embedded to create standing waves (in case of resonance) in the tube. Both the microphone and the speaker are connected to a computer, which sends a sine signal to the speaker and goes through a specified band of frequencies. The microphone will detect the amplitude of the signal, pass it to an attenuator to prevent an overdrive and the computer will visualize the recorded signal in form a spectrum, as seen in figure 8.

First of all we will determine the speed of sound by recording a spectrum in an aluminium tube of length $L = 8 \times 7.5 \text{ mm} = 600 \text{ mm}$ and using the distance of the resonance peaks to identify the linear law of the frequency differences between the resonance peaks.

Afterwards we will investigate the quality of the resonator and get a picture of how the measurements develop in time (reproducibility of measurements).

Experimental setup The tube resonator can be assembled from pieces of the length 5 cm or 7.5 cm. There are two end pieces, where one yields a speaker and the other a microphone. These end pieces are set up on a rail, so that the other tube pieces can easily be placed without any dislocation. The end pieces can be locked using a little bolt to prevent them from moving and therefore creating leaks in the resonator. The BNC splitter is connected to the SINE WAVE INPUT on the controller and the splitter is connected to the PC which operates as our sine wave generator. We plug the lead from speaker in the one lid to SPEAKER OUTPUT on the controller. Using a BNC cable the microphone in the other lid is connected to the MICROPHONE INPUT on the controller. Figure 7 yields a photo of the setup.



FIGURE 7: Photo of the experimental setup for the tube resonators. **Left:** A tube built from 8×7.5 cm pieces with a length of 600 mm. **Right:** A tube built from 2×7.5 cm pieces with a length of 150 mm.

2.1.1 Resonances of a Tube

We already discussed and derived the resonance condition in the theoretical backgrounds part. We found

$$2L = \frac{n}{\lambda} = n\frac{c}{\nu}, \qquad n \in \mathbb{N}.$$
(78)

2 Analysis

A spectrum for a 8×7.5 mm tube was recorded and we find equidistant resonance peaks. The spectrum is plotted in the figure 8.



FIGURE 8: Illustrated is the spectrum of a 8×7.5 mm tube in a range from 1000 Hz to 10000 Hz.

Using the fact that the peak intervals are equidistant, we can measure the speed of sound. Therefore we extract the frequency \hat{v} of every peak from the spectrum and assign an incremental index *i* to it. By indexing we need to take into account that the first peak we see in figure 8 is not the peak of index *i* = 1 but the one of *i* = 4, because three other peaks lie below 1 kHz and are hidden. A test measurement, beginning at 0 Hz, revealed three other peaks lying in the range from 0 Hz to 1 kHz. Now we fit a linear slope through resulting values, which is of the form

$$\mathcal{F}_c(\nu) = \frac{c}{2 \cdot 600 \,\mathrm{mm}} \nu,\tag{79}$$

where the adjustable parameter *c* represents the speed of sound in air. The fit procedure, visualized in figure 9, leads to $c = 344.028 \text{ m s}^{-1}$. The speed of sound in dry air at a given temperature ϑ in °C can be calculated by

$$c_{\rm air} = (331.5 + 0.6\,^{\circ}{\rm C}^{-1} \cdot \vartheta)\,{\rm m\,s}^{-1} = (331.5 + 0.6\,^{\circ}{\rm C}^{-1} \cdot 20.8\,^{\circ}{\rm C})\,{\rm m\,s}^{-1} = 343.98\,{\rm m\,s}^{-1}.$$
 (80)

This means a relative deviation of 0.014 % and an absolute deviation of 0.048 m s⁻¹ in this case. The perfect alignment of the values on the linear slope already suggests a very exact result.

2.1.2 High Resolution Spectrum of a Tube

A high resolution spectrum (sampling rate $\Delta v = 2$ Hz) of a 2 × 7.5 mm tube was recorded, which is illustrated in figure 10. To determine the quality of the resonator we need various parameters from the spectrum. These parameters, such as frequency, amplitude, width and phase are being extracted using a multi-peak Lorentz fit, which means in essence a sum of one Lorentz function for each peak.



FIGURE 9: Shown are the resonance frequencies of the 8×7.5 mm tube plotted over their ascending index number (starting at 1 in figure 8). Own can see that they are all pretty aligned on the fitted linear slope. The room temperature was measured as 20.8 °C.

As proposed in [4, p. 1-11] the spectrum can be described by a function of the form

$$A(\omega) = \left| \sum_{i=1}^{n} \frac{K_i e^{i\varphi_i}}{\omega_i^2 - \omega^2 + 2i\gamma_i \omega} \right|.$$
(81)

Because we are dealing with frequencies and not with angular velocities as this is the case for an oscillator we assume the fit function

$$\mathcal{F}(\nu) = \left| \sum_{i=4}^{11} \frac{K_i e^{i\varphi_i}}{(2\pi\nu_i)^2 - (2\pi\nu)^2 + 4\pi i\gamma_i\nu} \right|,\tag{82}$$

where K_i , φ_i , ν_i and γ_i are our adjustable parameters. The fit is produced by SpectrumSLC.exe, that came with the experimental setup.

In table 1 one can review the fit parameters of (82), which are the frequency v_i , the amplitude K_i , the phase φ_i and the width $\hat{\gamma}_i = \gamma_i/2\pi$. The *full width at half maximum* FWHM can be calculated with the formula

$$FWHM_i = \sqrt{3}\frac{\gamma_i}{2\pi} = \sqrt{3}\hat{\gamma}_i, \tag{83}$$

as proposed by [4, p. 1-12].

Quality of the Resonator The quality factor Q_i of the resonator can then be calculated using

$$Q_i = \frac{\nu_i}{\Delta \nu_i} = \frac{\nu_i}{\text{FWHM}_i}.$$
(84)

This so called Q-factor described the bandwidth of a given frequency in a resonator. Higher Q values indicate lesser rates of energy loss in dependency to the stored energy. The mean Q factor



FIGURE 10: Illustrated is a high-resolution spectrum (sampling rate 2 Hz) of a 2×7.5 mm tube in a range from 1000 Hz to 10 000 Hz. Additionally the fit is plotted, which was calculated by the Quantum Analogs programme.



FIGURE 11: Shown are the resonance frequencies of the 2×7.5 cm tube plotted over their wave number. One can see that they are all pretty aligned on the fitted linear slope.

in our setup was determined with

$$\langle Q \rangle \approx 140.$$
 (85)

Unfortunately the amplitude is given in an arbitrary unit, so that we can't compare the computed quality factor to the one of other resonators.

Index <i>i</i>	Frequency v_i	Amplitude K_i	Width $\hat{\gamma}_i$	Phase φ_i	FWHM _i	Quality Q_i
[1]	[Hz]	[a.u.]	[Hz]	[°]	[Hz]	[1]
1	1233.124	4.187	38.018	-128.549	65.849	18.727
2	2314.388	36.511	72.376	-110.150	125.359	18.462
3	3458.049	44.862	20.702	118.882	35.857	96.440
4	4579.261	27.414	20.067	-112.525	34.757	131.751
5	5714.161	59.303	17.094	-3.947	29.608	192.996
6	6844.003	47.580	18.176	19.222	31.482	217.396
7	7976.985	44.322	20.739	74.862	35.921	222.070
8	9105.760	43.360	23.321	134.527	40.393	225.428
					Mean	140.408

TABLE 1: Evaluation of the quality of the resonator. In the columns 2–5 the parameters of the fit function (82) are displayed. In the 6th column the FWHM has been computed from the fit parameters and the 7th column yields the corresponding quality factor.

Reproducibility of Measurements To evaluate the reproducibility of the measurements taken, two high resolution spectra were recorded in a distance of ≈ 10 min. The peak frequencies determined by fitting the curves can be reviewed in table 2. The shift $\Delta v_{12} = v_2 - v_1$ doesn't follow any particular rules. In the cutout in figure 12 one can see that the two spectra overlap nearly completely. This matches to the values in the table as the shift is for almost all peaks below 1 Hz.

Index <i>i</i> [1]	Frequency ν_1 [Hz]	Frequency ν_2 [Hz]	Difference Δv_{12} [Hz]
1	1233.124	1111.317	-121.807
2	2314.388	2315.499	1.111
3	3458.049	3457.938	-0.111
4	4579.261	4579.196	-0.065
5	5714.161	5714.137	-0.024
6	6844.003	6843.982	-0.021
7	7976.985	7976.992	0.007
8	9105.760	9105.792	0.032

TABLE 2: Evaluation of the reproducibility of the measurement. The first row can't be taken into account, as the fit for this peak doesn't converge very well, see figure 10.



FIGURE 12: To get a better view of the resonators quality we plotted two high-resolution spectra side-by-side. In the upper left corner we scoped out the peak near 5700 Hz. Because the spectra exactly overlap also at fine resolutions, like in the cut-out, an offset was added to the second measurement.

2.2 An Analogy for the Hydrogen Atom

Procedure In this part we want to determine the resonance frequencies for the spherical resonator and record spectra to identify the corresponding angular momentum quantum numbers.

First of all we will get an overview of how the spectrum behaves with regard to the orientation of microphone and speaker to each other. Second we will pick three resonances and observe how their amplitude behaves with changing angle between the two hemispheres; the resulting data will be fit to matching Legendre polynomials to identify the angular momentum quantum numbers. The Quantum Analogs programme offers an option *Measure wave function*, which allows to record the amplitudes of the resonances in dependency of the angle and plots it in a polar coordinate system. Last we will patch a ring of a certain thickness between the two hemispheres to break its symmetry, so that we can observe the degeneracy of the resonances.

Experimental setup The spherical resonator is built from two hemispheres, a lower and an upper one, which can be rotated with respect to each other. The rotation angle can be read off of a scale, printed on one of the hemispheres. As in the other experiments the BNC splitter is connected to the SINE WAVE INPUT on the controller and the splitter is connected to the PC which operates as our sine wave generator. We plug the lead from speaker in the lower hemisphere to SPEAKER OUTPUT on the controller. Using a BNC cable the microphone in the upper hemisphere is connected to the MICROPHONE INPUT on the controller. When observing the amplitude we also plug in the voltmeter to the DETECTOR OUTPUT jack. Figure 13 yields a photo of the setup.



FIGURE 13: Photo of the experimental setup for the spherical resonator. **Left**: The spherical resonator. **Right**: The spherical resonator with intermediate metal rings to induce the perturbation.

Remark concerning the polar angle ϑ When we read the angle α off the scale between the two hemispheres, we don't read the angle between microphone and speaker. This angle is the polar angle ϑ , which later appears as argument of the spherical harmonics. As seen in [4, p. 2-6] we can transform the plane angle α to the polar angle ϑ by using the identity

$$\vartheta = \arccos\left(\frac{1}{2}\cos\alpha - \frac{1}{2}\right).$$
(86)

A sketch showing the relation for the interesting part ($\alpha \in [0^\circ, 180^\circ]$) is plotted in figure 15.

2.2.1 Spectrum of the Spherical Resonator

To get an overview of how the resonator behaves when we rotate the two hemispheres we recorded several spectra for different angles α . In the resulting plot in figure 15 one can see that the amplitude is strongly angle dependent. This can easily be seen in the cutout, which



FIGURE 14: Sketch for the dependency of ϑ to α . See equation (86) for the mathematical expression.

covers a frequency range from 6000 Hz to 7500 Hz. One can also see that for opposing position of microphone and speaker ($\alpha = 180^{\circ}$) the peak amplitude reaches its maximum.

In figure 15 one can see smaller peaks between the main peaks above frequencies of 6 kHz. These are considered to be higher harmonics of the lower resonances and can be ignored. We will only care about the first three peaks anyway.

2.2.2 Determining the ℓ Quantum Numbers

We are about to determine the ℓ quantum numbers for the first three peaks from the left. Therefore we plug a voltmeter into the DETECTOR OUTPUT jack and set it to AC. The voltage passed from the microphone to the voltmeter is directly proportional to the amplitude. Because we cannot convert the voltage back to an amplitude due to the fact that we don't know the exact association between the two quantities, we will normalize these values to the interval [-1,1] by setting max $|A| \rightarrow \text{sign} (A) \cdot 1$.

Now we send the constant frequency to the speaker and read the voltage off the voltmeter. Then we turn the upper hemisphere by 5° and repeat the measurement. This process is repeated for all the angles $\alpha \in \{0^\circ, ..., 180^\circ\}$ in steps of 5° and for all peaks $i \in \{1, 2, 3\}$.

A problem of this method is, that the voltmeter cannot detect the sign of the voltage. To fix this issue we could either use an oscilloscope to visually observe when the voltage changes its sign or we could look at a graph of the resulting data and adjust it according to its appearance and change the sign manually whenever we think that the voltage just switched its sign.

Using the latter method and converting α to ϑ using the relation (86), the graph 16 was produced, where we fitted the first three associated Legendre polynomials according to the index of the peak that was investigated.

The data of the first peak matches best to the associated Legendre polynomial $P_{1,0}(\cos \vartheta)$ with an additional angular offset of 15.22°, which was introduced to compensate possible phase shifts induced by the experimenter. The mentioned fit with the angular offset is marked using a dashed line in the plots 16 and 17. One can see that the data looks like $\ell = 1$, but still it doesn't fit very



FIGURE 15: Overview for the spectrum of the spherical resonator for various rotation angles of the two hemispheres, in this case $\alpha \in \{0^{\circ}, 45^{\circ}, 90^{\circ}, 135^{\circ}, 180^{\circ}\}$. In the cutout in the range of 6000 Hz to 7500 Hz one can easily see that the amplitude depends on the angle.



FIGURE 16: Fitting of the associated Legendre polynomials to the measured resonances illustrated in figure 25 with the amplitude *A* as adjustable parameter. The dashed lines are also fits of the Legendre polynomials but with an angular offset v_0 as additional parameter.



well to the Legendre polynomial.

The data of the second peak already matches pretty good to $P_{2,0}(\cos \vartheta)$ without any phase shifts, so we can identify this peak as $\ell = 2$ with certainty. The phase which was still introduced was of the order 1.28°.

The third peak matches slightly better to $P_{3,0}(\cos \vartheta)$ than the first peak to $P_{1,0}(\cos \vartheta)$, so we cannot cleanly identify it as $\ell = 3$. Here the phase shift was 2.35°.

For a discussion of the phase shift and possible errors, see section 3.

To get a better view of the affiliation of the peaks to any quantum numbers we plot the data and the fitted, associated Legendre polynomials in a polar axis. Therefore we additionally plot the negative part of the fit to get the complete projection of the spherical harmonic on the \mathbb{R}^2 plane. In figure 17 one can now see that the fitted, associated Legendre polynomials reproduce the well known spherical harmonic for $\ell \in \{1, 2, 3\}$ very well.



FIGURE 17: Polar representation of the fitting of the associated Legendre polynomials illustrated in figure 16 with the amplitude *A* as adjustable parameter for the solid curves; the dashed lines are also fits of the Legendre polynomials but with an angular offset v_0 as additional parameter. In the polar plot also the negative parts of the fitting functions were drawn, so one can easily identify the $\ell = 1$, $\ell = 2$ and $\ell = 3$ resonance as they look completely similar to the *normal* spherical harmonics.

2.2.3 Measuring the Wave Function

With the Quantum Analogs programme it is possible to measure the amplitude automatically with dependency on the angle α . This function is named *Measure Wave Function* and it displays a polar plot of the amplitudes similar to this displayed in figure 17. To measure these, the experimenter has to go through all angles α in steps of 10°; the button *Measure value* sends the last selected frequency to the microphone and measures the amplitude. The checkbox *Complete symmetry* projects the measured part from the range $\vartheta \in \{90^\circ, \ldots, 180^\circ\}$ to the rest of the polar grid, such that the spherical harmonic is completed. Using this function, it is much easier to identify the ℓ quantum numbers of the resonances. As the degeneracy wasn't lifted yet using a ring to perturbate the system we will only observe m = 0 modes. Due to the fact that this degeneracy exists and all $m \neq 0$ modes will also be excited we may encounter interference.

In the following figure the measured wave function will be drawn on the left side, while the adjacent plot yields the spherical harmonic with the best fit. Differences to the normal spherical harmonics occur for the following frequencies

- 2.3 kHz, figure 18: The nodal plane is widened in contrast to the computer model. The resonance looks more like $\ell = 2$, but as the legs are deformed it is more likely that a $m \neq 0$ -mode is also excited and interferes. We characterize the resonance as $\ell = 1$.
- 3.7 kHz, figure 19: The legs split up, again probably due to a *m* ≠ 0 resonance that interferes. Nevertheless the characteristics lets us identify this resonance as *l* = 2.
- 4.9 kHz, figure 20: This resonance looks all like $\ell = 3$.
- 6.2 kHz, figure 21: At first sight this resonance has certain attributes of *l* = 5, because of the eight minor legs. On closer examination we find that neither the angle nor the amplitude matches the *l* = 5 resonance. Therefore the split up of the lobe in the nodal plane also has to be of interfering emergence. Therefore the resonance is classified as *l* = 4.
- 7.4 kHz, figure 22: This resonance looks all like $\ell = 5$.
- 8.6 kHz, figure 23: Again a split up of the inner legs can be seen, which would match to *ℓ* = 7. Although the angle and the amplitude of the inner legs don't fit at all to *ℓ* = 7. Therefore we assume it to be *ℓ* = 6.
- 9.6 kHz, figure 24: Although the edges look a little bit awkward, the resonance is identified as ℓ = 7.

In the figures 18 to 24 one can see, that for 90° the signal is nearly always distorted. This is due to the fact, that at this angle $m \neq 0$ modes have their amplitude maximum.

In figure 25 the ℓ quantum numbers were placed next to the corresponding peak. The unlabeled peaks are resonances of higher *n* quantum numbers. Because the polar amplitudes for these were very noisy we didn't try to identify their ℓ quantum numbers.

2.2.4 Breaking the Symmetry

To break the symmetry of the resonator, thin rings were inserted between the two hemispheres, where the rings could be combined to achieve spacings between the two hemispheres of $\delta \in \{0 \text{ mm}, 3 \text{ mm}, 6 \text{ mm}, 9 \text{ mm}\}$.

As the symmetry axis no longer goes through the microphone, the $m \neq 0$ modes will be excited as well. In figure 26 one can already see, that the position of the peaks is δ dependent.

In figure 26 the range $\nu \in [1 \text{ kHz}, 5.5 \text{ kHz}]$ is displayed. As we know from the last part of the experiment, the peaks have been identified with their corresponding ℓ quantum number. It is placed next to the peak. In the plot one can see that with increasing δ the distance between the peaks increases. For $\ell = 1$ this is investigated in detail in figure 27.

For $\ell = \{2, 3\}$ one can observe that for bigger δ more peaks appear. This is due to the fact that there occurs a split up for the *m* modes and for these ℓ there are more *m* modes, because they have to follow the rule

$$m\in\{-\ell,\ldots,0,\ldots,\ell\}.$$

Here the sign can't be distinguished, so the model reduces to $m \in \{0, ..., \ell\}$.

Let's look at $\delta = 9$ mm: For $\ell = 2$ the peaks $m \in \{0, 1, 2\}$ can be observed in figure 26 where the m = 1 peak is very diffuse. For $\ell = 3$ one should be able to observe peaks for $m \in \{0, 1, 2, 3\}$, but the modes m = 1 and m = 2 overlap and smear out each other. Also the m = 3 peak is too small to identify it correctly.

$2|_{\mathsf{Analysis}}$



Figure 24: $\ell=7$ at $\nu=9.6\,kHz$



FIGURE 25: Overview for the spectrum of the spherical resonator, where the in figure 16 in detail observed resonances were fitted. The analysed resonances are marked with their corresponding angular momentum quantum number. The unlabelled peaks are resonances of n = 2 and are not discussed any further.



FIGURE 26: Cutout of the spectrum of the spherical resonator in dependency of the perturbation induced by a ring inserted between the hemispheres, where the thickness of the ring was varied between $\delta \in \{0 \text{ mm}, 3 \text{ mm}, 6 \text{ mm}, 9 \text{ mm}\}$. Here one can see that with increasing thickness of the ring the split-up between the peaks increases. In the picture we plotted the $\ell = 1$, $\ell = 2$ and $\ell = 3$ resonances. The curves have been smoothed.



FIGURE 27: Cutout of the spectrum of the spherical resonator in dependency of the perturbation induced by a ring inserted between the hemispheres, where the thickness of the ring was varied between $\delta \in \{0 \text{ mm}, 3 \text{ mm}, 6 \text{ mm}, 9 \text{ mm}\}$. For the $\ell = 1$ and $\ell = 2$ resonances the split-up of the peak is illustrated.

For $\ell = 1$ we expect the split up $m \in \{-1, 0, 1\}$, which can be observed as $m \in \{0, 1\}$, because of the indistinguishable sign. This aspect assorts well with figure 27 where exactly this can be observed.

If we plot the peak distance over the ring thickness (namely the perturbation strength), one can see that the correlation is approximately linear. To verify this, a linear function \mathcal{F}_a of the form

$$\mathcal{F}_a(\delta) = a \cdot \delta \tag{87}$$

was fitted, although the plot suggests a right bended curve. It turned out that the slope a = 19.1905. Therefore a differential split up of

$$\frac{\mathrm{d}\Delta\nu}{\mathrm{d}\delta} = \frac{\mathrm{d}\mathcal{F}_a(\delta)}{\mathrm{d}\delta} = a \approx 19\,\mathrm{Hz\,mm^{-1}} \tag{88}$$

can be assumed.

As a quantum mechanical analogy one could name the *Zeeman-effect* whose split up of the energy levels is also linear if the perturbation is handled in first order perturbation theory.

$$\Delta E = g\mu_B B \tag{89}$$

where *g* is the gyromagnetic factor. For further information see [1, p. 360].

If we compare the classical and the quantum mechanical case, one can say, that the thickness of the spacer-ring δ is an analogy to the strength of the magnetic field *B*. Larger δ means larger *B*.

2.3 Miming the H₂ Molcule

Procedure In this part we are going to discuss the spectrum of two coupled spherical resonators. We will cast an eye at the split up of the orbitals in dependency to the coupling strength (the iris diameter β).

2 Analysis



FIGURE 28: Plotted is the peak distance for $\ell = 1$ in Hz on the *y*-axis over the ring thickness in mm on *x*. The fit is linear which does not really reproduce the real behaviour of the peak distance as the real curve seems to be bended right. This is due to the fact that the linear behaviour was derived using perturbation theory, which is only an approximation.

Experimental setup The spherical resonator is built from two hemispheres, a lower and an upper one, which can be rotated with respect to each other. The rotation angle can be read off of a scale, printed on one of the hemispheres. As in the other experiments the BNC splitter is connected to the SINE WAVE INPUT on the controller and the splitter is connected to the PC which operates as our sine wave generator. We plug the lead from speaker in the lower hemisphere to SPEAKER OUTPUT on the controller. Using a BNC cable the microphone in the upper hemisphere is connected to the MICROPHONE INPUT on the controller. When observing the amplitude we also plug in the voltmeter to the DETECTOR OUTPUT jack. Figure 29 yields a photo of the setup.



FIGURE 29: Photo of the experimental setup for the coupled spherical resonators



2.3.1 Analysing the Spectrum

In figure 30 one can see the increasing split up with increasing coupling strength, in this case the iris diameter. It is also visible that for very weak coupling ($\beta = 5 \text{ mm}$) the two coupled resonators nearly behave like two uncoupled ones. Compare therefore also in figure 27 the red line ($\delta = 0 \text{ mm}$). The split up of the peak groupings in dependency of the iris diameter appears to be linear.

The resonances have been annotated with their corresponding orbitals in figure 30. These resonaces have been associated by comapring the measured data to a reference spectrum of H_2 .

Resonances with a variation of the angle α on the upper sphere with a constant iris diameter of $\beta = 25$ mm are shown in figure 31. The spectrum is analogous to the second energy level of the molecule. We see a change in the amplitude of the π -resonances because the corresponding orbital only has a symmetry plane that lies in between the two spheres. The σ -orbitals have a symmetry axis in the direction of the line connecting midpoints of each sphere. The rotation of the microphone on the resonator has the same symmetry axis.

2.4 Modelling a Solid State Body

Procedure We analyse the spectrum of a tube that is built from strung together tube pieces and iridae in a periodic fashion. This models the periodic potential of a crystal lattice. The dispersion relation of the tube can be taken from the occurring resonance frequencies. We do this for several combinations of iridae with varying diameter and tube pieces of varying length. After that we analyse the model of a superlattice and the effect of a defect in the lattice.

Experimental Setup The modified tube resonator is built from pieces of the length 5 cm or 7.5 cm and iridae with diameters of 10 mm, 13 mm or 16 mm that are placed on a rail. To create a periodic structure the iridae and pieces are placed in an alternating order. One of the end pieces has an inbuilt microphone while the other end piece has the speaker. The cable of the speaker is connected to the SPEAKER OUTPUT and the microphone cable to the MICROPHONE INPUT on the controller. We use the PC sound card as the sine wave generator again and connect the BNC cable from the sound card output with the SINE WAVE INPUT. A possible setup for the structure with a defect in the lattice is shown in figure 32.

2.4.1 Dependency on the Lattice Constant

First we compare the spectra of two tubes built with the same number of tube pieces, where one has tube pieces of the length 5 cm and the other of the length 7.5 cm. In both tubes the iridae have diameters of 16 mm. The dispersion relation of each tube is shown in figures 33 and 34 in the extended and reduced zone scheme. The wave numbers k are given by the boundary constraint

$$k = n \frac{\pi}{L} \tag{90}$$

where *L* is the length of the tube and *n* the index of the *n*-th resonance. Since there are no negative frequencies of an acoustic wave we use the same resonance frequency of given wave number for the respective negative wave number.

Both dispersion relations show a clear difference to that of the unperturbed tube in figure 11 since band gaps appear after the 7th and every following 8th wave numbers. But the linear fit in the case of the unperturbed tube also appears in the tubes with iridae as an enveloping function. Similar wave numbers for both modified tubes also occur at resonance frequencies of similar magnitude. A free electron and an electron in a periodic potential show the same relation for their respective dispersion relations with the difference, that the dispersion relation of a free electron is a quadratic function.



FIGURE 30: Resonances of the H₂ molecule mimic in dependency to the coupling strength (in this case the iris diameter β). One can see that the stronger the coupling (larger β), the sharper the split up of the *molecular* orbitals. All spectra were recorded with $\alpha = 0^{\circ}$.



FIGURE 31: Resonances of the H₂ molecule mimic in dependency of the angle α of the scale on the upper sphere. All spectra were recorded with a constant iris diameter of $\beta = 25$ mm. Only the π resonances are dependent upon the angle as the symmetry axis of the σ peaks is the *z*-axis with origin at the iris.

2 Analysis



FIGURE 32: Setup of the tube resonator for the modeling of solid state body. In this case a lattice defect is inserted.

In the figures we see that the wave numbers of each band belong to a definite Brillouin zone. The wave number with -

$$k = \frac{\pi}{a} \tag{91}$$

marks the border of the first Brillouin zone of a lattice with the lattice constant a. This is reproduced in the acoustic case.



FIGURE 33: Dispersion relation of the tube built from 8×5.0 cm pieces with intermittent iridae 7×16 mm. The border of the Brillouin zones is marked with solid lines, while the middle is marked with a dashed line. The band gaps are highlighted using a grey box.

2.4.2 Dependency on the Number of Elementary Cells

In the previous section we saw that there are eight resonance frequencies between two band gaps (except for the first Brillouin zone where there are seven) when we use eight tube pieces. The tube pieces are analogous to the unit cells of a crystal lattice. According to theory the number



FIGURE 34: Dispersion relation of the tube built from 8×7.5 cm pieces with intermittent iridae 7×16 mm. The border of the Brillouin zones is marked with solid lines, while the middle is marked with a dashed line. The band gaps are highlighted using a grey box.

of possible wave vectors for each energy band is proportional to the number of unit cells in the lattice.

Figure 35 shows that the number of resonances in a band is indeed proportional to the number of tube pieces. Only for the first Brillouin zone this is not the case as the resonance at $k = 0 \text{ cm}^{-1}$ can't be reproduced in an acoustic experiment. By reading the diagram from bottom to top we see that the solid is constructed from the elementary cells in accordance to the tight binding theory.

2.4.3 Dependency on the Iris Diameter

The iridae in the structure are analogous to the atomic cores in the lattice. By varying the diameter $\delta \in \{10 \text{ mm}, 13 \text{ mm}, 16 \text{ mm}\}$ of every iris we change the potential energy at each core in the analogy. A smaller iris diameter corresponds to a stronger binding, i.e. a bigger potential. This alters the band structure of the solid as seen in figures 36 and 37. The peaks in the spectrum become less defined for decreasing iris diameter. Also the band gaps become bigger since the first resonance frequency in each band remains approximately the same. Looking at it from the other way around the band gap decreases for increasing iris diameter until it disappears when the diameter of iridae and tube pieces are the same. Here we see that the theory of the periodic potential applies as an explanation of the band gap.

2.4.4 Consequences of a Superlattice

A superlattice is created by changing the diameter of every second lattice in the resonator. We did this for iridae with diameters of 16 mm and 13 mm and tube pieces of length 5 cm. The first Brillouin can be determined using the formula

$$k = \frac{\pi}{2L}n\tag{92}$$



FIGURE 35: Dependency of the number of peaks on the number of elementary cells. A $n \times 5.0$ cm resonator was used. One can easily see, that for every additional elementary cell we gain an additional peak in every band.



FIGURE 36: Dependency of the number of peaks on the iris diameter. A 8×7.5 cm resonator was used with a variing iris of the diameter $\delta \in \{10 \text{ mm}, 13 \text{ mm}, 16 \text{ mm}\}$. The peak distance depends on the diameter of the iris. This compression of the dispersion relation leads to a bigger band gap.



FIGURE 37: Dispersion relation of the tube built from 8×7.5 cm pieces with intermittent iridae 7×16 mm in comparison with the dispersion of the same tube with 7×13 mm iridae. The border of the Brillouin zones is marked with solid lines, while the middle is marked with a dashed line. The band gaps are highlighted using a grey box.



The spectrum of this resonator is depicted in figure 38 and the dispersion relation in figure 39. An additional band gap appears for a wave number at the middle of a half of the Brioullin zone. This is due to the unit cell of the superlattice being double the size of a tube piece which results in an altered periodicity. The number of peaks in each band remains the same but they aren't distributed evenly on the two 'sub bands'. By introducing smaller iridae into the structure, the original band gaps become sligthly bigger. This effect was also seen in the analysis of different iris diameters.



FIGURE 38: Given in this figure is the overview spectrum for the resonances of a resonantor with a superlattice, where the resonator itself is built up out of 12×5.0 cm tubes with alternating iridae of $\delta = 13$ mm and 16 mm.

2.4.5 Inserting a defect

A defect can be brought into the resonator by either changing the length of one of the tube pieces or the diameter of one iris. In both cases we use a resonator built from 12 unit cells. Figure 40 shows spectra for the case of the defect cell. The regular cells are modeled by the 5.0 cm long pieces while the defect cell is a 7.5 cm long piece. Appart from a slight variance of the resonance frequencies there appears to be an additional resonance within the first band gap. Counting the number of the resonance peaks we see that the additional peak is actually the resonance belonging to the 12th unit cell. Its gap to the other peaks of the first band shifts with the position *n* of the defect. Up to n = 6 and 7 this gap increases while for higher *n* it decreases again. This symmetry stems from the symmetry of the tube, since the spectrum should be the same if we swap the end pieces containing the speaker and microphone. The additional state in the band is similar to the additional band gap of the superlattice. The gap should also appear for higher energy bands, but the amplitude of the resonance can't be detected in the spectrum.

Spectra for the case of a defect iris are shown in figure 41. The regular iridae in this structure are of diameter 16 mm and the defect iris at position n of diameter 10 mm. As in the case of a defect cell, the spectra of positions up to n = 6 are mirrored for positions of greater n. The defect again increases the gap between certain peaks. We see that the gaps around the n-th peaks in the bands, counting up from the lowest and down from the highest frequency of a band, are widened



FIGURE 39: Dispersion relation of a tube built from 12×5.0 cm pieces with a superlattice of alternating iridae of $\delta = 13$ mm and 16 mm. The border of the Brillouin zones is marked with solid lines, while the middle is marked with a dashed line. The band gaps are highlighted using a grey box. One can also see that the superlattice causes a split up of the bands themselves. These additional band gaps are also marked.

compared to the other resonances of the band. At n = 6 an actual superlattice from two unit cells with 6 tube pieces is created as 5 periodical band gaps appear. For the other cases the widening of the gaps comes from a superlattice with broken symmetry.

2 Analysis



FIGURE 40: Spectra of a resonator with 12 unit cells and a defect cell at the *n*-th position. The resonator was built from 11×5.0 cm pieces and one 7.5 cm piece as the defect cell. Apart from a slight shift of the resonance frequencies there also appears an additional resonance within the first band gap.



FIGURE 41: Spectra of a resonator with 12 unit cells and a defect iris at the *n*-th position. The resonator was built from 10 iridae of diameter 16 mm and one iris of diameter 10 mm as the defect. As a result of the defect the resonance frequencies of the n-th peaks counting from top and bottom of each band show a greater gap towards the other peaks in the band. At n = 6 the defect creates a super lattice structure with 6 tube pieces in each unit cell. 40

3 Error Discussion

First of all some global error sources need to be taken into account, which apply for all experiments.

- 1. The apparatuses need to be used in the linear region, as too high amplitudes lead to overdrive at the microphone. However too low amplitudes would lead to cross talk phenomena.
- 2. In case of cross talk we might get additional peaks, which are not part of the spectrum of the resonator being measured. This is due to a too high amplitude, hence a too low value at the attenuator.
- 3. The combination of microphone and speaker has a resonance, which was still taken into account for measurements, because it is likely that this resonance of the device shadows other resonances.
- 4. The resolution with which the spectra were recorded is finite. Normal spectra were recorded at $\Delta \nu = 10$ Hz while high resolution spectra were recorded at $\Delta \nu = 2$ Hz. According to the Nyquist-Shannon sampling theorem we can locate a peak only to a certainty of $\Delta \nu/2$.

3.1 Tube Resonator as a Model for the Infinite Potential Well

In the first part several values were calculated. In the following we will iterate an error discussion.

1. The speed of sound: In figure 9 a linear slope was fitted through the peaks of the spectrum of a 8×7.5 cm tube. The fit function was given by

$$\mathcal{F}_c(n) = \frac{c}{2L}n\tag{93}$$

The fitting parameter *c* was determined as 344.028 m s^{-1} . gnuplot determined an asymptotic standard error of $c_{\text{err}} = 0.146657 \text{ m s}^{-1}$. This makes up a standard deviation of Var(c) = 0.021508.

2. Due to the lack of any comparable data, we were not able to calculate or even estimate an error for the quality factor *Q*, which was determined as

$$\langle Q \rangle \approx 140$$
 (94)

Another point is that the quality of the resonator is very dependent to the height of the peaks (the amplitude), see table 1. As the FWHM is dependent to the amplitude of the peak the question arises, if the quality might vary with different amplitudes.

3.2 An Analogy for the Hydrogen Atom

The second part yields a very qualitative analysis of the spherical resonator.

- 1. In the unperturbed case we were working under the assumption, that $m \neq 0$ modes aren't excited. Although the spherical resonator is not a perfect sphere and therefore also these modes are weakly excited. This means that the m = 0 modes get deformed a little, see also figures 18 to 24.
- The microphone is not punctate. Therefore it covers a certain range of solid angles. Furthermore the angle is measured in steps of 1°.
- 3. In figure 28 we postulated a linear dependence of the split up to the perturbation (ring thickness). This assumption is based on only four values.

3 Error Discussion

3.3 Modelling a Solid

In the third part very qualitative analysis was performed for the tube resonators as a model for a solid.

- 1. All the peaks for the dispersion relations were picked out by hand and maybe some peaks were determined a slightly wrong position.
- 2. The vertical resolution (frequency) of the dispersion relations is determined by the measurement resolution of the associated spectrum.
- 3. In higher bands the resonances in the bands were weakened a lot, so distinguishing to peaks was very difficult. Invisible resonances were counted, so that no index shift occurs and the dispersion relation doesn't get deformed.
- 4. Disturbances in the spectrum lead to harder identification of the resonances, especially in the lower bands. Therefore some resonances needed to be extrapolated.
- 5. To calculate the wave number k there formula

$$k = \frac{n\pi}{L} \tag{95}$$

where *L* is the length of the resonator which was computed using $L = N \cdot a$. The length of the iridae was not taken into account in this calculation, which leads to an error in the overall length. It is not clear how the iridae affect the length of the resonator due to their little diameter.

4 Summary

4 Summary

4.1 Tube Resonator as a Model for the Infinite Potential Well

For an empty 8×7.5 cm tube equidistant peaks were observed. Using these peaks to speed of sound was determined as

$$c = 344.028 \,\mathrm{m \, s^{-1} \pm 0.146\,657 \,m \, s^{-1}}$$
 at 20.8 °C (96)

in contrast to the literature value of c = 343.98 °C at 20.8 °C. One can see that the result is pretty exact.

Next in line was the determination of the quality factor Q of a 2 × 7.5 cm tube resonator. Two high resolution spectra were recorded and fitted using the fit function

$$\mathcal{F}(\nu) = \left| \sum_{i=4}^{11} \frac{K_i e^{i\varphi_i}}{(2\pi\nu_i)^2 - (2\pi\nu)^2 + 4\pi i\gamma_i\nu} \right|$$
(97)

The quality factor was determined with a mean value of

$$\langle Q \rangle \approx 140$$
 (98)

The analysis of the reproducibility of measurements shows that in a time distance of ≈ 10 min the spectra are nearly perfectly reproducible.

4.2 An Analogy for the Hydrogen Atom

The recorded spectra showed a dependency to the angle between the two hemispheres. For the resonances $\ell \in \{1, 2, 3\}$ the associated Legendre polynomials were fitted and plotted in a polar axis to illustrate the spherical harmonics.

The analysis for the polar amplitude allowed us to identify the resonances $\ell = 1, ..., 7$ in the spectrum for n = 1. The various polar plots were partially a little deformed as $m \neq 0$ modes were excited as well.

The *m* degeneracy was lifted using spacer rings. A nearly linear dependence of frequency split up and perturbation strength (ring thickness) was discovered. The $\pm m$ degeneracy couldn't be lifted.

4.3 Coupled Spherical Resonators as a Model of the Hydrogen Molecule

We find, that the hydrogen molecule can be reproduced by a coupling of two spherical resonators by an iris. The diameter of the iris simulates the coupling strength, which increases for increasing diameter. This results in a greater split up of the resonances belonging to individual molecular orbitals. At the smallest iris diameter the resonances look almost like those of the single spherical resonator. The σ and π orbitals can be identified by changing the position of the microphone along the scale of the angle α . Because of symmetry reasons, the amplitude of the π peaks changes for different angles while that of the σ peaks remains the same.

4.4 Tube Resonator with Iridae as a Model of a 1d Solid State Body

By introducing iridae at each connection of tube pieces in the tube resonator, we create an analogy to the periodic potential of a one dimensional solid. The dispersion relation shows a band gap structure enveloped by a linear function that correponds to the dispersion relation of the unperturbed tube. The band gaps appear at wave numbers that mark the border of a Brillouin zone. The width of a Briollin zone changes with the length of the tube pieces. A variation of the

4 | Summary

iris diameter changes the size of the band gaps. The number of unit cells is equal to the number of resonances within a band. We see a combination of these effects for a superlattice structure built by an alternating use of two different iridae. There also appears an additional band gap within each brillouin zone of the superlattice. A defect in the lattice can be simulated by either a different sized unit cell or an iris with deviating diamter exchanged for any of the regular elements of the lattice. A lattice with a defect can be classified as a pseudo superlattice as the defect results in similar effects as in the case of the superlattice compared to the regular lattice.

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